REFERENCE SOLUTION OF PROBLEM 44

Lemma. *If S is stationary and C is a club, then $S \cap C$ is stationary.*

Proof. Let D be another club. Since $C \cap D$ is again club, notice:

$$(S \cap C) \cap D = S \cap (C \cap D) \neq \emptyset.$$

Hence $S \cap C$ intersects every club. □

Problem 44. *Suppose $S \subseteq \omega_1$ is stationary. Show:*

(a) *There are arbitrarily large $\alpha \in S$ with $\sup(S \cap \alpha) = \alpha$.*

(b) *For every limit ordinal $\alpha < \omega_1$, let C_α denote the set of limit ordinals $\gamma < \omega_1$ such that for all $\beta < \gamma$ there is a closed set C with $(*)$:*

  (i) $C \cap \beta = \emptyset$,

  (ii) $\mathrm{otp}(C) = \alpha + 1$,

  (iii) $\max(C) = \gamma$, *and*

  (iv) $C \setminus \{\gamma\} \subseteq S$.

Show that each C_α contains a club w.r.t. ω_1 and conclude that for any $\alpha < \omega_1$ there is a closed $C \subseteq S$ with otp$(C) = \alpha + 1$.

Proof. Let $S \subseteq \omega_1$ be stationary.

(a) Assume this is false. Then there is an upper bound, i.e. some $\gamma \in \omega_1$ such that for all $\alpha \geq \gamma$, $\sup(S \cap \alpha) < \alpha$.

Then the function $f : S \setminus \alpha \to \omega_1$, $f(\alpha) = \sup(S \cap \alpha)$ is regressive. Hence, by Fodor's Lemma, there is a stationary, in particular unbounded, set $T \subseteq S \setminus \alpha$ such that f is constant on

T. Let $f[T] = \{\beta\}$. Since T is unbounded, there are $\varepsilon, \delta \in T$ with $\varepsilon > \delta > \beta$. Now notice that $\delta \in S \cap \varepsilon$, hence

$$\beta = f(\varepsilon) = \sup(S \cap \varepsilon) \geq \delta > \beta \lightning.$$

(b) We perform an induction on the limit ordinals α below ω_1.

The base case is $\alpha = \omega$. Check that $C_\omega = \{\gamma \in \omega_1 \cap \mathrm{Lim} \mid \gamma \text{ is a limit point of } S\}$. (Note that this is *not* the derivation of S, as it may contain $\gamma \notin S$): Let γ be a limit point of S and $\beta < \gamma$ and show that β satisfies $(*)$:

Take $(\gamma_n)_{n<\omega}$ to be some strictly increasing sequence above β in S that converges to γ. Then consider the clearly closed set $C := \{\gamma_n \mid n < \omega\} \cup \{\gamma\}$. Check the four properties in $(*)$:

(i) $C \cap \beta = \emptyset$, because γ_n runs above β,
(ii) $\mathrm{otp}(C) = \omega + 1$, since $(\gamma_n)_{n<\omega}$ has ordertype ω and we add one,
(iii) $\max(C) = \gamma$, obviously, and
(iv) $C \setminus \{\gamma\} \subseteq S$ since the γ_n run inside S.

Furthermore, C_ω is unbounded by (a) and closed, as limit points of limit points of S are again limit points of S.

The successor step is $\alpha \to \alpha + \omega$. By induction, let $D \subseteq C_\alpha$ be a club. Define a stationary set (by the lemma) $S_0 := S \cap D$. Define $E := \{\gamma \in \mathrm{Lim} \cap \omega_1 \mid \gamma \text{ is a limit point of } S_0\}$. As before, E is club. Now show that $E \subseteq C_{\alpha+\omega}$.

Let $\delta \in E$ and $\beta < \delta$. Show that δ satisfies $(*)$. δ is a limit point of S_0, so there is a $\gamma \in S_0$ with $\beta < \gamma < \delta$. Then $\gamma \in S \cap D \subseteq S \cap C_\alpha \subseteq C_\alpha$. By $(*)$ there is a closed F with $F \cap \beta = \emptyset$, $\max(F) = \gamma$, $\mathrm{otp}(F) = \alpha + 1$ and $F \setminus \{\gamma\} \subseteq S$. Then, since $\gamma \in S$, $F \subseteq S$.

Now choose a strictly increasing sequence $(\delta_n)_{n<\omega}$ in S above γ that converges to δ. Consider $C = F \cup \{\delta_n \mid n \in \omega\} \cup \{\delta\}$. Note that this union is disjoint. Check the properties in $(*)$:

(i) $C \cap \beta = \emptyset$, because δ_n runs above $\gamma > \beta$,
(ii) $\mathrm{otp}(C) = \alpha + \omega + 1$, since $\mathrm{otp}(F) = \alpha$ and we add $\omega + 1$,
(iii) $\max(C) = \delta$, obviously, and
(iv) $C \setminus \{\delta\} \subseteq S$ since the δ_n run inside S and $F \subseteq S$.

For the limit case suppose α is a limit of limit ordinals and suppose for all limits $\nu < \alpha$ there is a club in C_ν. Define the following:

- Choose a strictly increasing sequence $(\alpha_n)_{n<\omega}$ of limit ordinals, except $\alpha_0 = 0$, that converges to α.
- Choose $(\beta_n)_{n<\omega}$ such that $\alpha_n + \beta_n = \alpha_{n+1}$ for all $n < \omega$, in particular $\beta_0 = \alpha_1$. Note that the β_n are all limits smaller α.
- For all $n < \omega$ let F_n be a club contained in C_{β_n} (by induction).
- Define $S_0 := S \cap \bigcap_{n<\omega} F_n$ and notice that it is stationary, since $\omega < \mathrm{cof}(\omega_1)$ and hence by a previous exercise $\bigcap_{n<\omega} F_n$ is club.
- Define $E := \{\gamma \in \mathrm{Lim} \cap \omega_1 \mid \gamma \text{ is a limit point of } S_0\}$ and notice that it is club by the same arguments as before.

Now show that $E \subseteq C_\alpha$. Let $\gamma \in E$, $\beta < \gamma$. Show that β satisfies $(*)$:

Choose a sequence $(\gamma_n)_{n<\omega}$ above β in S_0 that converges to γ and satisfies $\forall n < \omega : \gamma_n + 1 < \gamma_{n+1}$. Notice that for all $n < \omega$, $\gamma_{n+1} \in F_n$ and hence by $(*)$ there is a closed D_n with

$D_n \cap (\gamma_n + 1) = \emptyset$, $\max(D_n) = \gamma_{n+1}$, $D_n \setminus \{\gamma_{n+1}\} \subseteq S$ and $\mathrm{otp}(D_n) = \beta_n + 1$.

Define $C = \bigcup_{n<\omega} D_n \cup \{\gamma\}$. For all $n < \omega$, $\max(D_n) = \gamma_{n+1}$ and $\min(D_{n+1}) > \gamma_{n+1}$. Hence C is a disjoint union and therefore by the choice of the β_n:

$\mathrm{otp}(C) = (\sum_{n<\omega} \mathrm{otp}(D_n)) + 1 = (\sum_{n<\omega} \beta_n + 1) + 1 = (\alpha_1 + \sum_{0<n<\omega}(1 + \beta_n)) + 1 = (\alpha_1 + \sum_{0<n<\omega} \beta_n) + 1 = \alpha + 1$.

C is clearly closed. Then check $(*)$:

(i) $C \cap \beta = \emptyset$, because γ_n runs above β and all the D_n are above γ_n,
(ii) $\mathrm{otp}(C) = \alpha + 1$ (see above),
(iii) $\max(C) = \gamma$, obviously, and
(iv) $C \setminus \{\gamma\} \subseteq S$ since all the D_n lie in S.

This concludes the first part of the proof.

Now show that for all $\alpha < \omega_1$ there is a closed $C \subseteq S$ with $\mathrm{otp}(C) = \alpha + 1$. Perform an induction over α. If $\alpha = 0$, any singleton out of S will do.

$\alpha \to \alpha + 1$. Let $C \subset S$ be closed with ordertype $\alpha + 1$. $|C| \leq |\alpha + 1| \leq \omega$ and ω_1 is regular, hence C is not unbounded in ω_1. So choose some $\gamma \in S \setminus (\sup C + 1)$. Notice that $C \cup \{\gamma\}$ is still closed and has ordertype $\alpha + 1 + 1$.

$\alpha \in \mathrm{Lim}$: C_α contains a club, so $C_\alpha \cap S$ is not empty. Then we can take $\gamma < \omega_1$, $\gamma \in C_\alpha \cap S$. Take some $\beta < \gamma$ and a closed set C such that $C \setminus \{\gamma\} \subseteq S$ and $\mathrm{otp}(C) = \alpha + 1$. Since $\gamma \in S$, $C \subseteq S$.

$\square$